Indian Statistical Institute, Bangalore
M. Math. Second Year Second Semester
Operator Theory

Mid-term Examination Date: 17-02-2017
Maximum marks: 100 Time: 3 hours

(1) Let $\mathcal{H}$ be a Hilbert space of dimension greater than 1. Show that the Banach space $\mathcal{B}(\mathcal{H})$ of bounded operators on $\mathcal{H}$, with operator norm, is not a Hilbert space. [15]

(2) Let T be a normal operator on a Hilbert space. Show that $\lambda \in \mathbb{C}$ is an eigenvalue of T (i.e., $Tx = \lambda x$ for some $x \neq 0$) if and only if $\bar{\lambda}$ is an eigenvalue of T^*. Show that eigen-vectors corresponding to distinct eigenvalues of T are orthogonal. [15]

(3) Let P be an orthogonal projection on a Hilbert space $\mathcal{H}$. Show that the spectrum of P is contained in $\{0, 1\}$. [10]

(4) Show that the unilateral shift on l^2 is not compact. [10]

(5) Let $\mathcal{A}$ be a commutative unital Banach algebra. Define the Gelfand dual $\hat{\mathcal{A}}$ of $\mathcal{A}$. Show that maximal ideals of $\mathcal{A}$ are in one to one correspondence with $\hat{\mathcal{A}}$. [20]

(6) Let $\mathcal{A}$ be a unital Banach algebra and let $\rho(x)$ be the resolvent set for an element x of $\mathcal{A}$. Let ϕ a continuous linear functional $\mathcal{A}$. Show that the map $\lambda \mapsto \phi((x-\lambda)^{-1})$ is a complex differentiable function on $\rho(x)$. [20]

(7) Let $\mathcal{A}, \mathcal{B}$ be unital C^*-algebras. Let $\pi : \mathcal{A} \to \mathcal{B}$ be a unital $*$-homomorphism. Show that $\sigma(\pi(a)) \subseteq \sigma(a)$ for $a \in \mathcal{A}$. Using this or otherwise show that $\|\pi(a)\| \leq \|a\| \ \forall a \in \mathcal{A}$. [20]